Heat flow on plane GAF

Ching Wei Ho

Joint work with Brian Hall, Jonas Jalowy and Zakhar Kabluchko

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The heat flow operator

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1. The heat flow operator $e^{-\frac{\tau}{2}\frac{d^2}{dz^2}}$ on an entire function F is defined by the power series

$$e^{-\frac{\tau}{2}\frac{d^2}{dz^2}}F(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{\tau}{2}\right)^n \frac{d^{2n}}{dz^{2n}}F(z)$$

   for τ ranges in a disk on the complex plane.
2. If F is a polynomial, this power series terminates.
3. In general, the entire function F satisfies a certain growth rate.
4. We also write $F(\tau, z) = e^{-\frac{\tau}{2}\frac{d^2}{dz^2}}F(z)$. It satisfies the PDE

$$\frac{\partial F}{\partial \tau} = -\frac{1}{2}\frac{\partial^2 F}{\partial z^2}.$$

Example 1 of the heat flow

1. Take $F(z) = z^k$.
2. The heat flow of F is the Hermite polynomial (with "variance" τ) of the same degree

$$F(\tau, z) = e^{-\frac{\tau}{2}\frac{d^2}{dz^2}} z^n = \tau^{n/2} H_n\left(\frac{z}{\sqrt{\tau}}\right).$$

Example 2 of the heat flow

1. Example: $P_N(z) = (z-1)^{N/2}(z+1)^{N/2}$.
2. By [Kabluchko, 2022], the root distribution of

$$e^{-\frac{t}{2N}\frac{d^2}{dz^2}} P_N$$

converges to the same limiting eigenvalue distriburtion as the random matrix

$$\begin{pmatrix} I_{N/2} & 0 \\ 0 & -I_{N/2} \end{pmatrix} + \text{GUE}.$$

3. The large-N behavior of heat flow at time t/N connects to random matrix theoery and free probability theory.

The main questions

Main Question

Consider the plane GAF G.

1. *How does G evolve under the heat flow operator*

$$\exp\left(-\frac{\tau}{2}\frac{d^2}{dz^2}\right)?$$

2. *What can we say about the evolution of zeros of G?*

1. These questions are motivated by a consideration similar to the previous example, but with initial distribution on the complex plane, instead of on the real line.

Heat flow on the plane GAF

The plane GAF

1. Define the plane GAF (or simply GAF) by

$$G(z) = \sum_{k=0}^{\infty} \xi_k \frac{z^k}{\sqrt{k!}}$$

   where ξ_k are independent complex Gaussian random variables.
2. Fact: for every $\varepsilon > 0$, $|G(z)| \leq C_\varepsilon \exp\left((\frac{1}{2} + \varepsilon)|z|^2\right)$ a.s.
3. This means G is a.s. of order 2 and of type $1/2$.

Well-definedness of heat flow

Theorem (Hall–H.–Jalowy–Kabluchko, 2023)

Let $\tau \in \mathbb{C}$ such that $|\tau| < 1$. The heat flow operator on G is well-defined a.s. and can be computed as the following.

1. $e^{-\tau \frac{\partial^2}{\partial z^2}} G(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{\tau}{2} \frac{\partial^2}{\partial z^2} \right)^k G(z)$
2. *If $\tau = |\tau| e^{i\theta}$, then*

$$e^{-\tau \partial^2 / \partial z^2} G(z) = \frac{1}{\sqrt{2\pi|\tau|}} \int_{\mathbb{R}} G(i e^{i\theta/2} x) e^{-\frac{(-ie^{-i\theta/2} z - x)^2}{2|\tau|}} dx.$$

- I omit two other ways to compute the heat flow of G.

Distribution of zeros

1. Distribution of zeros of G is approximately the roots of Weyl polynomial (thus, the distribution of zero is "uniform" with spacing of order 1).
2. What is the distribution of zeros of $e^{-\frac{\tau}{2}\frac{\partial^2}{\partial z^2}}G$?

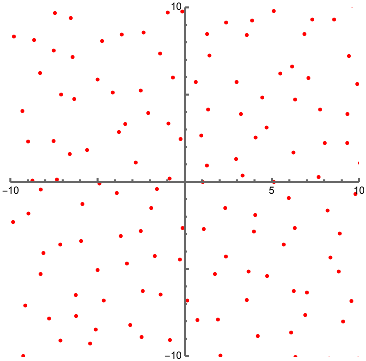

Figure: Zeros of G

Evolution of the distribution of zeros

Theorem (Hall–H.–Jalowy–Kabluchko, 2023)

$$\frac{\mathcal{Z}\left(e^{-\frac{\tau}{2}\frac{\partial^2}{\partial z^2}}G\right)}{\sqrt{1-|\tau|^2}} \stackrel{d}{=} \mathcal{Z}(G).$$

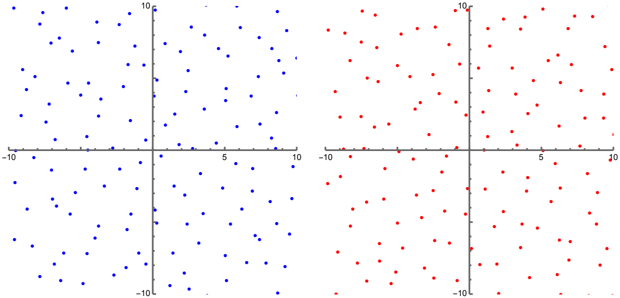

Figure: LHS: Rescaled zeros of $e^{-\frac{\tau}{2}\frac{\partial^2}{\partial z^2}}G$; RHS: Zeros of G

Reason of the evolution of distribution

1. Define the random holomorphic function $V_\tau G$ by

$$(V_\tau G)(z) = (1-|\tau|^2)^{1/4} e^{\bar{\tau} z^2/2} \left(e^{-\frac{\tau}{2}\frac{\partial^2}{\partial z^2}} G \right) \left(z\sqrt{1-|\tau|^2} \right).$$

2. $\{V_\tau G(z)\}_z$ has the same distribution as $\{G(z)\}_z$. That is, $V_\tau G$ is also a GAF.
3. Since $(1-|\tau|^2)^{1/4} e^{\bar{\tau} z^2/2}$ has no zero,

$$\frac{\mathcal{Z}\left(e^{-\frac{\tau}{2}\frac{\partial^2}{\partial z^2}} G \right)}{\sqrt{1-|\tau|^2}} = \mathcal{Z}(V_\tau G) \overset{d}{=} \mathcal{Z}(G).$$

Evolution of zeros

1. That the distribution is invariant up to a scaling does not tell you how the zeros evolve!
2. How do the zeros evolve under heat flow?

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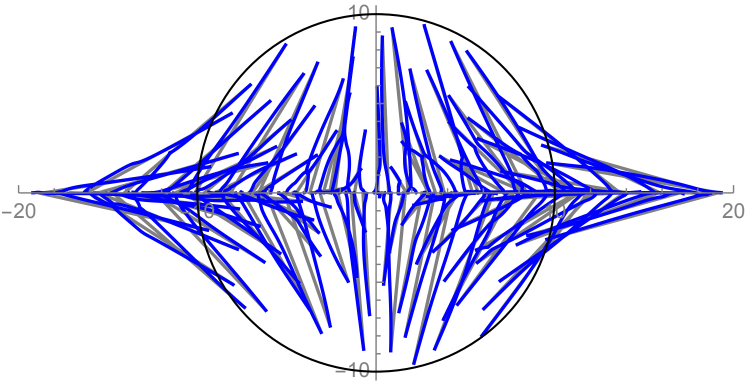

Figure: Zero evolutions: plot each zero a with gray lines $a + \tau\bar{a}$. $0 \leq \tau \leq 1$.

Evolution of zeros

1. The zeros evolve approximately along the gray lines $a + \tau\bar{a}$.
2. Want to understand the error

$$z(\tau) - (a + \tau\bar{a}).$$

Evolution of a zero

1. The notation $z^b(\cdot)$ means the random holomorphic function

$$\left(e^{-\tau\frac{\partial^2}{\partial z^2}}G\right)(z^b(\tau)) = 0$$

when we condition on $G(b) = 0$ ($z^b(0) = b$).

2. Under this notation, $z^b(\cdot)$ is defined in a disk with random radius.

Theorem (Hall–H.–Jalowy–Kabluchko, 2023)

$$z^a(\tau) \stackrel{d}{=} a + \tau\bar{a} + z^0(\tau)$$

Evolution of zeros

1. Want to understand the error

$$z^a(\tau) - (a + \tau\bar{a})$$

which has the same distribution as $z^0(\tau)$.

2. The next simultation computes this error starting at a; that is

$$a + [z^a(\tau) - (a + \tau\bar{a})].$$

Evolution of a zero

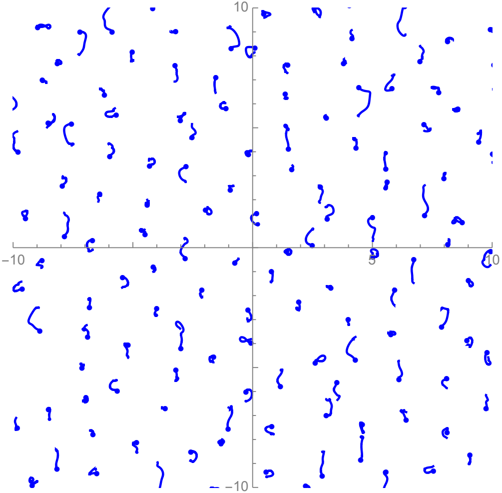

Figure: Simulation of $z^a(\tau) - \tau\overline{z^a(0)} \overset{d}{=} a + z^0(\tau)$.

A relation to $SU(1,1)$

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1. Recall that for all $|\tau| < 1$,

$$(V_\tau G)(z) = (1 - |\tau|^2)^{1/4} e^{\bar{\tau} z^2/2} \left(e^{-\frac{\tau}{2} \frac{\partial^2}{\partial z^2}} G \right) \left(z\sqrt{1 - |\tau|^2} \right)$$

   is also a GAF.

2. Can obtain this result by the metaplectic representation of $SU(1,1)$ on the Hilbert space $HL^2(\mathbb{C})$ of entire functions F satisfying

$$\int_C |F(w)|^2 \frac{e^{-|w|^2}}{\pi} d^2 z < \infty,$$

   called the Segal–Bargmann space.

A relation to $SU(1, 1)$

1. The functions
$$\frac{z^n}{\sqrt{n!}}, \quad n \geq 0$$
form an orthonormal basis of the Segal–Bargmann space $HL^2(\mathbb{C})$.
2. The GAF
$$G(z) = \sum_{n=0}^{\infty} \xi_n \frac{z^n}{\sqrt{n!}}$$
can be heuristically thought to be Gaussian distributed on $HL^2(\mathbb{C})$.
3. Thus, heuristically, given any unitary operator U on $HL^2(\mathbb{C})$, $U(G)$ is again a GAF.

A relation to $SU(1,1)$

1. Given an $A \in SU(1,1)$ of the form
$$A = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix},$$
define a projective unitary representation on the Segal–Bargmann space $HL^2(\mathbb{C})$ by
$$V(A)f(z) = \pm \frac{1}{\sqrt{p}} \int_{\mathbb{C}} \exp\left(\frac{1}{2}\frac{\bar{q}}{p} z^2 - \frac{1}{2}\frac{q}{p}\bar{w}^2 + \frac{1}{p} z\bar{w} \right) f(w) \frac{e^{-|w|^2}}{\pi} \, d^2w.$$
2. Then $(V_\tau f)(z) = (1-|\tau|^2)^{1/4} e^{\bar{\tau}z^2/2} \left(e^{-\frac{\tau}{2}\frac{\partial^2}{\partial z^2}} f \right) \left(z\sqrt{1-|\tau|^2} \right)$ can be obtained by
$$V_\tau f = V(A_\tau) f$$
where
$$A_\tau = \frac{1}{\sqrt{1-|\tau|^2}} \begin{pmatrix} 1 & \tau \\ \bar{\tau} & 1 \end{pmatrix}.$$